# AN OPTIMAL CONTROL APPROACH TO THE IRRIGATION PLANNING PROBLEM


Sofia O. Lopes*, Fernando A. C. C. Fontes†, M. d. Rosário de Pinho†, Rui M. S. Pereira*, A. Manuela N. Golçalves*, Gaspar J. Machado*

* *Universidade do Minho, Portugal*

† *Universidade do Porto, Portugal*

Emails: `sofialopes@math.uminho.pt, faf@fe.up.pt, mrpinho@fe.up.pt, rmp@math.uminho.pt, mneves@math.uminho.pt, gjm@math.uminho.pt`



**Abstract**— We study the irrigation systems as an optimal control problem, where the trajectory is the water in the soil and the control is the amount of water introduced in the soil via irrigation. We consider two problems: first what we call the Yearly planning problem to determine the amount of water spent along the year and then the Initial planning problem to calculate the minimum amount of water in order to fulfill the culture needs given an initial amount of water at the beginning of a year. Since none of these models are exact representations of reality, especially when uncertainty, like weather, is involved, predictive control techniques are used to replan our problem.



**Keywords**— optimal control; nonlinear programming; irrigation systems.


## 1 Introduction

Climate change has put enormous stress in many regions of the world, namely the Iberian Peninsula (Naim Haie, 2008) (Naim Haie, 2011). According to the 2011 National Plan for the Efficient Use of Water (*Uso da água na agricultura 2011*, 2011), water demand in Portugal is approximately $7500 \times 10^6\, m^3/year$, in the three sectors: urban, agricultural and industrial. The agricultural sector is, in terms of volume, the largest consumer (80 %), where the wastes are estimated at 37,5 %. We want to optimize the water use in the irrigation of farm fields by means of an optimal control problem. The model takes into account the evapotranspiration, percolation and runoff. For such problem the trajectory is the water in the soil and the control is the amount of water introduced in the soil via its irrigation system. Thus, we want to minimize the volume of water used in irrigation, knowing that the variation of water in the soil is given by the hydrologic balance equation. The rainfall is predicted taking into account historical data for the location and a multiple linear regression model.

For an atypical year (i.e., when our rainfall model differs completely from reality) the results obtained by our model may not be the best. For example, we may face the situation where the quantity of water needed for irrigation may greatly exceed the computed optimal solution of our problem. In this case the culture can died and cause major economic and environmental damage. Because of this, we develop a new model based on replan: first we calculate the optimal solution based on the previous model and then at every time step we recalculate a new dynamic based on real data.

## 2 Model

We want to minimize the planning of water resources to be used for irrigation of a farm field. We choose an optimal control formulation where the state (or trajectory) is the amount of water in the soil and the control is the amount of water added to the soil by irrigation systems. Our problem is then the following:

$$
\begin{array}{ll}
\min & \sum_{i=1}^{N-1} u_i \\
\text{subject to:} & \\
x_{i+1} = x_i + \Delta f\left(t_i, x_i, u_i\right) & \text{a.e. } i = 1 : N-1 \\
x_i \geq x_{\min} & i = 1 : N \\
u_i \geq 0 & \text{a.e. } i = 1 : N-1 \\
(x_1, x_N) \in C, &
\end{array}
$$

where $x = (x_1, \ldots, x_N)$ is the trajectory, $u = (u_1, \ldots, u_{N-1})$ is the control, $f$ is balance hydrologic function, $x_{\min}$ is the hydrological need of the crop, $C$ is a given set, $\Delta$ is the time step discretization and $N = 12/\Delta$. Observe that $x_i \geq x_m in$ is a **state constraint**. Note that for the sake of simplicity we will consider a unit area farm field.

The dynamic equation that represents the hydrologic balance, is given by

$$
\begin{aligned}
f(t_i, x_i, u_i) = u_i &+ \text{rainfall}(t_i) \\
&- \text{evapotranspiration}(t_i) \\
&- \text{losses}(x_i),
\end{aligned} \tag{1}
$$

where the evapotranspiration is the evaporation of the soil and the transpiration of the crop, and the losses are the losses of water due to the runoff and deep infiltration.

As for the endpoint state constraint $(x_1, x_N) \in C$, two different problems arise depending on $C$:

**Yearly Planning:** the initial state (1st of January) is the same as the final state (1st of January of the next year),

$$(x_1, x_N) \in C \Leftrightarrow C = \{(x_1, x_N) : \ x_1 = x_N\}.$$

**Initial Planning:** the initial state is fixed and the final one is free,

$$(x_1, x_N) \in C \Leftrightarrow \\ C = \{(x_i, x_N) : x_1 = X_{initial}, x_N \in \mathcal{R}^1\}$$

### 2.1 *Rainfall models*

In order to estimate rainfall we use the monthly rainfall data from *Instituto Português do Mar e Atmosfera*, in the Lisbon area. We define an average (using the 10 years data) rainfall for each month of the year, the *rain monthly average* is :$10^{-3}\times$

| J | F | M | A | M | J |
|---|---|---|---|---|---|
| 111.4 | 94.7 | 80.2 | 57.1 | 29.62 | 18.84 |

| J | A | S | O | N | D |
|---|---|---|---|---|---|
| 1.26 | 7.04 | 30.6 | 127 | 121.98 | 119.3 |

($m^3$/month)

We consider two models of rainfall: one based on the average monthly rainfall of the last 10 years in Lisbon area and another taking into account the best linear combination of average monthly rainfall from the last 10 years and the amount of rainfall in the previous month. These models are:

$$\text{rainfall}(1)(t_i) = \textit{precipitation factor} \\ \times \textit{rain monthly average}(t_i), \quad (2)$$

$$rainfall(2)(t_i) = \textit{precipitation factor} \\ \times (c_2 \textit{rain monthly average}(t_i) \quad (3) \\ + c_1 rainfall(t_{i-1})) + \epsilon.$$

To take into account different weather scenarios, the models are multiplied by a *precipitation factor*. This factor, allows us to consider a typical year if this factor is 1, a drought year if it is less that 1 and a rainy year if it is above 1.

In (S.Lopes, 2013), it is shown that the rainfall model (3.2) is statistically acceptable.

### 2.2 *Evapotranspiration model*

We use the Pennman - Monteith methodology (I. A. Walter, 2002) to evaluate the evapotranspiration of our culture (potatoes) along the year. In this case we consider:

$$ET(t_i) = K_c ET_0(t_i),$$

where $K_c = 0.825$ is the culture coefficient for the evapotranspiration (in our case potatoes) and $ET_0$ is the tabulated reference value of evapotranspiration given in (Raposo, 1996) for the Lisbon region. The evapotranspiration of our culture in Lisbon is given by the following table: $10^{-3}\times$

| J | F | M | A | M | J |
|---|---|---|---|---|---|
| 19.8 | 28.05 | 55.275 | 89.1 | 116.325 | 137.775 |

| J | A | S | O | N | D |
|---|---|---|---|---|---|
| 155.925 | 136.95 | 84.975 | 53.625 | 22.275 | 16.5 |

($m^3$/month)

### 2.3 *Modeling "losses" of water*

Our model of infiltration is based on the postulate of Horton's equation that says that infiltration decreases exponentially with time (Horton, 1940). That means the dynamical equation is

$$x_{i+1} = x_i + h(g(t_i, u_i) - \beta x_i), \quad (4)$$

where $g(t_i, u_i) = u_i + rainfall(t_i) - evapotranspiration(t_i)$.

From (1) and (4), one may say $losses(t_i) = \beta x(t_i)$, where $\beta$ depends on the type of soil.

## 3 Results

Our study is based on the fact that we have monthly data in a field of potatoes in the region of Lisbon with unit area. Thus our parameters are:

$$x_{\min} = 0.56/12\,\Delta\ \mathrm{m}^3 \\ x_0 = 4\Delta x_{\min}\,\mathrm{m}^3 \\ h = 1 \\ \beta = 15\%.$$

To obtain the numerical solution for optimal control problems we approximate the problems by a sequence of finite dimensional of nonlinear programming problems, see (Betts, 1943) and (Mangasarian, 1969). To implement this optimization problem we use *fmincon* function of MatLab with the algorithm "active set", by default. Although it is a local search method, the convexity of the problem allows us to conclude that the solution obtained is the global solution.

### 3.1 *Yearly planning*

Here, the initial state and the final state are equal. These types of problems are useful if we intend to predict the amount of water spent along the year. Imagine we want to "build" a tank or reservoir that will be used to irrigate our field during the year. We intend to optimize the construction of this tank so that our field is properly irrigated. In this case we use equation (2) to model the rainfall. Taking all this into account the results obtained are shown in Figure 1.

Average Year

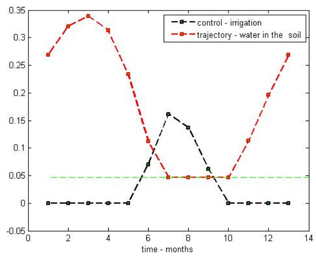


Figure 1: Precipitation Factor = 1.

Note that, in all figures the green line is the hydrological need of the crop.

It can be seen that the values of the optimal amount of water in the soil at the beginning and at the end are the same — $0.26m^3$, and that it will stay at the minimum allowed value from July till October. Regarding the irrigation, it should start in May, the maximum value is in July and stops in October.

The water needs for the whole year is given by the value of the objective function at the end of the simulation which was 0.4301 ($\mathrm{m}^3$/month).

Note that in order to save as much water as possible - for an hypothetical drought year - and supposing the owner of the field has enough space and money to build a tank of water that will fulfill his needs we may take *precipitation factor* = 0.5. Doing so one obtains the results in Figure 2.

Drought Year

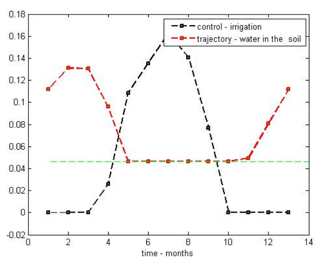


Figure 2: Precipitation Factor = 0.5.

The values of the optimal amount of water in the soil at the beginning and at the end are the same, 0.11, and that it stays at the minimum allowed value from May till November. Regarding the irrigation, it should start in March, the maximum value is in July and stops in October. The water needs for the whole year are given by the value of the objective function at the end of the simulation which is 0.6490 ($\mathrm{m}^3$/month).

### 3.2 *Initial planning*

Now we intend to use the minimum amount o water so as to fulfill the culture needs given a certain initial amount of water in the soil in the beginning of a year and using the rainfall model . For all examples shown next this value is $4x_{\min}$ $\mathrm{m}^3$/month.

We compare the solutions obtained using our model of rainfall with solutions obtained having a prior knowledge of the rainfall in the years 2008 and 2010. As mentioned before, we use data from *Instituto Português do Mar e Atmosfera* (www.meteo.pt).

Year 2008

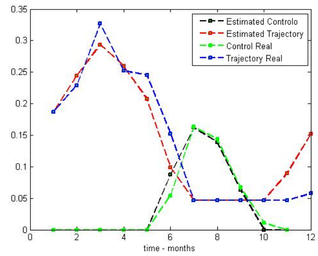


Figure 3: Using new model for the year 2008

According to our model,
in the year 2008:
- the estimated water needs is 0.4512 $\mathrm{m}^3$/year;
- the water needs is 0.4386 $\mathrm{m}^3$/year.

Year 2010

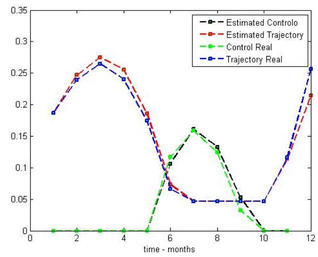


Figure 4: Using new model for the year 2010

in the year 2010:
- the estimated water needs is 0.4524 $\mathrm{m}^3$/year;
- the water needs is 0.4319 $\mathrm{m}^3$/year

## 4 Replan

Due to the unpredictability of weather conditions, the general model presented in the previous section may not describe accurately the system. If we have an atypical year, values obtained by the rainfall model may be completely different from reality. This means, there is a high probability that the results obtained by this model may not be the best. To overcome this drawback, we develop a new model based on replaning. Firstly, we determine the optimal solution based on the (OCP) and then, at every time step, we recalculate a new dynamic based on real data, where the *rainfall* is a table with real data. We test the replan model for the last ten years and we observed that state constraint was violated in the years 2003, 2005, 2007, 2008 and 2009. For instance considering the year 2009, the result obtained is described in figure 5:

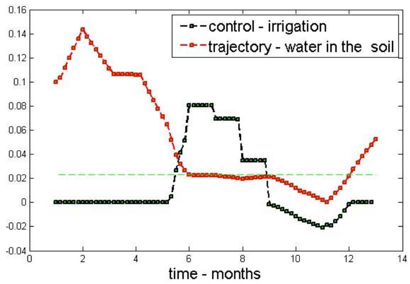


Figure 5: Results for the replan planning for the year 2008.

Due to the fact of using real data, the state constraint may be violated. To prevent this situation, a new model had to be considered. In this new model, we use soft state constraints instead of hard state constraint. Therefore, we consider an optimal control problem with a penalization in the cost function, as follows:

$$\begin{aligned} \min \quad & \sum_{i=1}^{N-1} u_i + KW(x_i) & \\ \text{s.t.:} \quad & x_{i+1} = x_i + \Delta f(t_i, x_i, u_i), & \text{a.e. } i = 1, \ .., \ N-1 \\ & u_i \geq 0, & \text{a.e. } i = 1, \ .., \ N-1 \\ & x_1 = X_{initial}. & \end{aligned}$$

where the function $W$ takes the value zero if the *state constraint** is not violated and increases exponential otherwise, where this *state constraint** has a safety margin.

Optimal Solution with Replan in 2008

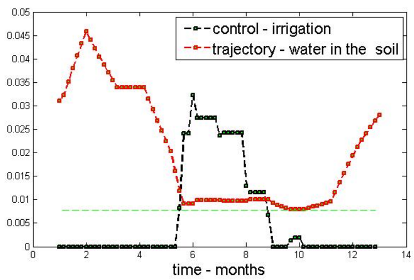


Figure 6: Optimal Solution is 0.4418 m$^3$/year.

Optimal Solution Known a Prior the Rainfall in 2008

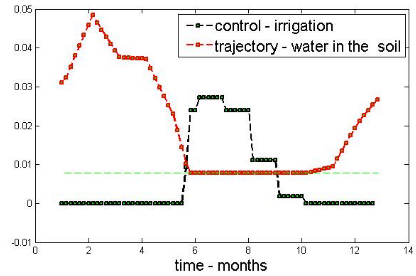


Figure 7: Optimal Solution is 0.4416 m$^3$/year.

We can see in Figure 6 and Figure 7 that for the year 2008 the state constraint is not violated and the result is very close to the optimal solution obtained knowing a priori the rainfall. The same was observed for the remaining years. We also test this model for atypical years, where the rainfall is equal to [90 90 90 0 0 0 0 0 0 0 0 0]./1000 m$^3$.

Previous Model

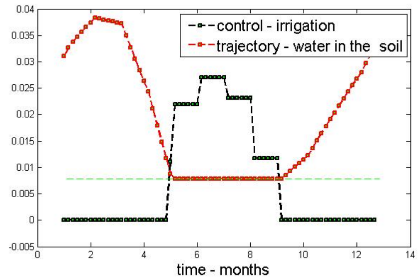


Figure 8: Optimal Solution is 0.5143 m$^3$/year.

Replan

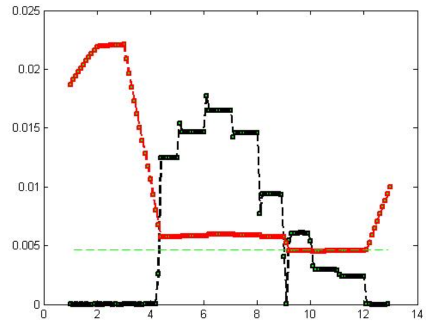

Figure 9: Optimal Solution is 0.7425 m$^3$/year.

From Figure 8, Figure 9, and Figure 10, we can conclude that the previous model fail completely from September to December, however the

Rainfall Known a Prior

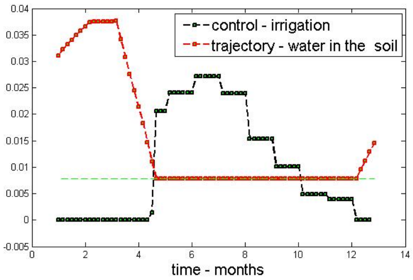


Figure 10: Optimal Solution is 0.7210 $m^3$/year.

replan gives a good answer. On the other hand, we can see in the next figures that the safety margin can be reduce using small discretization steps.

Optimal Solution with Replan in 2008

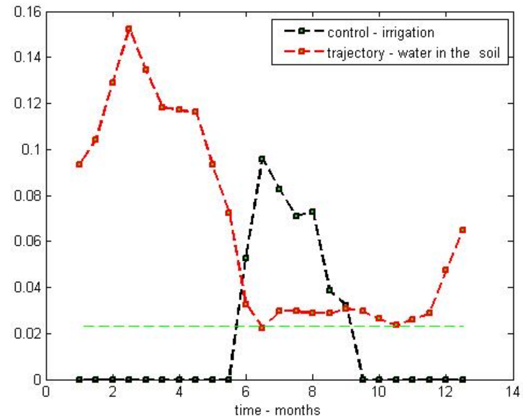


Figure 11: N=24

Optimal Solution with Replan in 2008

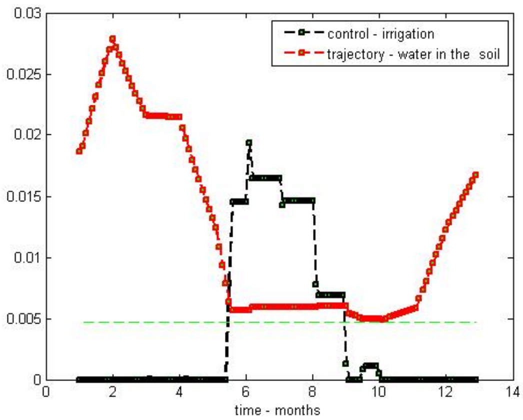


Figure 12: N=120

## 5 Conclusion

The obtained results are consistent with what it was expected. In drought years the irrigation needs are greater and start earlier in the year. The irrigation needs in the case of the first example are consistent with the irrigation necessary for potatoes in the Lisbon area (Raposo, 1996) and (Pereira, 2004).

We can conclude that the new model that we present in this paper is an improvement for atypical years when comparing with the model presented in (S.Lopes, 2013). The predictive control techniques combined with a penalization of cost function allows us to overcome the obstacles that have arisen with the planning of the irrigation problem under real data. Using small discretization steps reduce the bad behavior of the trajectory.

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